# Chiral polytopes whose smallest regular cover is a polytope

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#### Abstract

We give a criterion for when the smallest regular cover of a chiral polytope  $\mathcal{P}$  is itself a polytope, using only information about the facets and vertex-figures of  $\mathcal{P}$ .

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#### 1 Introduction

Abstract polytopes are combinatorial objects that generalize convex polytopes. Every abstract 3-polytope can be identified with a face-to-face tiling of a closed surface, though some such tilings do not yield abstract 3-polytopes. More generally, an abstract n-polytope can be thought of as a collection of abstract (n-1)-polytopes glued together in a 'nice' way that ensures that the resulting structure still resembles the incidence relation of a convex polytope.

The study of symmetry is central in the theory of abstract polytopes. The most symmetric polytopes are called *regular*, and there is a well-developed theory of regular polytopes; see [9] for an extensive overview. Polytopes that have full symmetry under abstract rotations but no symmetry by an abstract reflection are called *chiral*. Chiral polytopes have also been the subject of much study, but they have proven much more difficult to pin down than regular polytopes. Chiral polytopes were introduced in [14], building on earlier work of Coxeter and others on chiral maps and honeycombs. Since then, many papers have investigated the structure of chiral polytopes; see [1, 2, 6, 13] for a broad selection.

Every chiral polytope has a unique smallest regular polytope-like object that covers it. Our goal is to determine when the smallest regular cover of a chiral polytope is itself a polytope. (This is essentially Problem 3 in [13].) A partial answer is given in [11, Cor.

7.5], covering the case where the facets or vertex-figures are regular. Here we will give an answer for the case where the facets and vertex-figures are both chiral. Furthermore, our characterization only uses information about the facets and vertex-figures; we do not need to know any of the global structure of the chiral polytope in question.

# 2 Background

### 2.1 Polytopes

The basic notions of abstract polytopes can be found in [9] and chiral polytopes are described in [14]. Here we review the necessary background.

Suppose  $\mathcal{P}$  is a partially-ordered set with a unique minimal element  $F_{-1}$  and a unique maximal element  $F_n$ . If the maximal chains of  $\mathcal{P}$  all have the same length, then  $\mathcal{P}$  is a graded poset, where each element has a rank equal to 1 more than the maximum rank of all elements it covers. In our treatment, we give the minimal element a rank of -1. If  $\mathcal{P}$  also satisfies the two conditions below, it is an (abstract) n-polytope:

- (a) (Diamond condition): Whenever F < G and rank(G) rank(F) = 2, there are exactly two elements H such that F < H < G.
- (b) (Strong connectivity): Suppose F < G and  $\operatorname{rank}(G) \operatorname{rank}(F) \ge 3$ . Then the poset interval  $(F, G) := \{H : F < H < G\}$  has a connected Hasse diagram.

From now on, we will understand 'polytope' to mean 'abstract polytope'.

Each element F of rank n-1 in an n-polytope induces an (n-1)-polytope; namely, the poset  $[F_{-1}, F] := \{H : F_{-1} \le H \le F\}$ . We refer to this as a facet of  $\mathcal{P}$ . (In some contexts, 'facet' can also refer to the element F itself, but at present we will not need to refer to those elements separately.) Similarly, if v is an element of rank 0, then the vertex-figure at v is the poset  $[v, F_n]$ , which is also an (n-1)-polytope. Finally, a medial section of  $\mathcal{P}$  is a facet of a vertex-figure of  $\mathcal{P}$  (or indeed, a vertex-figure of a facet).

The maximal chains of a polytope are referred to as *flags*. If two flags differ only in their face of rank i, then we say that the flags are i-adjacent, or simply adjacent. For every  $i \in \{0, \ldots, n-1\}$ , each flag has a unique i-adjacent flag.

We say that  $\mathcal{P}$  covers  $\mathcal{Q}$  if there is a function  $\pi: \mathcal{P} \to \mathcal{Q}$  that preserves the partial order, the rank of each face, and such that whenever two flags of  $\mathcal{P}$  are *i*-adjacent, so are their images under  $\pi$ .

An automorphism of  $\mathcal{P}$  is a bijection from  $\mathcal{P}$  to itself that preserves rank and the partial order. We denote the automorphism group of  $\mathcal{P}$  by  $\Gamma(\mathcal{P})$ . There is a natural action of  $\Gamma(\mathcal{P})$  on the flags of  $\mathcal{P}$ , and if this action is transitive, then  $\mathcal{P}$  is called regular. In this case, the automorphism group of  $\mathcal{P}$  is a quotient of the string Coxeter group

$$W_n = \langle r_0, \dots, r_{n-1} \mid r_i^2 = 1 \text{ for all } i, (r_i r_j)^2 = 1 \text{ for all } |i - j| > 1 \rangle.$$

A regular polytope is called *orientable* (or *directly regular*) if every relator in  $\Gamma(\mathcal{P})$  has even length in the generators  $r_i$ . In this case, the *rotation subgroup* of the automorphism group,

which consists of the words of even length in  $\Gamma(\mathcal{P})$ , is a subgroup of index 2 in  $\Gamma(\mathcal{P})$ . We denote this group by  $\Gamma^+(\mathcal{P})$ .

A polytope  $\mathcal{P}$  is *chiral* if the action of  $\Gamma(\mathcal{P})$  on the flags of  $\mathcal{P}$  has two orbits such that adjacent flags always lie in different orbits. The facets of a chiral polytope are all isomorphic to some polytope  $\mathcal{K}$  that is either chiral or directly regular, and the same is true of the vertex-figures. The facets of the facets (and vertex-figures of the vertex-figures) must be directly regular (see [14, Prop. 9]). When working with chiral polytopes, we assume that we have chosen a *base flag*; the form of the automorphism group for a chiral polytope depends on the choice of base flag.

# 2.2 String $C^+$ -groups

For each  $n \geq 1$ , let

$$W_n^+ = \langle s_1, \dots, s_{n-1} \mid (s_i \cdots s_j)^2 = 1 \text{ for all } i < j \rangle.$$

Consider a group  $\Gamma = W_n^+/M = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ , where each  $\sigma_i$  is the image of  $s_i$  under the natural projection. We define:

$$\Gamma_0 = \langle \sigma_2, \dots, \sigma_{n-1} \rangle$$

$$\Gamma_{n-1} = \langle \sigma_1, \dots, \sigma_{n-2} \rangle$$

$$\Gamma_{0,n-1} = \langle \sigma_2, \dots, \sigma_{n-2} \rangle.$$

The group  $\Gamma$  is called a *string*  $C^+$ -group if it satisfies a certain *intersection condition* (see [14, Prop. 7]), but the following recursive definition is equivalent (see for example [5, Proposition 3.13]).

**Definition 2.1.**  $\Gamma = W_n^+/M$  is a string  $C^+$ -group if:

- n=1 and  $|\Gamma|>1$ , or
- $\Gamma_0$  and  $\Gamma_{n-1}$  are both string  $C^+$ -groups and  $\Gamma_0 \cap \Gamma_{n-1} = \Gamma_{0,n-1}$ .

The group  $W_n^+$  has an outer automorphism that sends each  $s_i$  to  $\overline{s_i}$  defined by:

$$\overline{s_1} = s_1^{-1}$$

$$\overline{s_2} = s_1^2 s_2$$

$$\overline{s_i} = s_i \text{ for } i > 2.$$

Indeed, we can identify  $W_n^+$  with the even-word subgroup of  $W_n$  by identifying each  $s_i$  with  $r_{i-1}r_i$ , and the outer automorphism of  $W_n^+$  is just conjugation by  $r_0$  in  $W_n$ . Given a quotient  $\Gamma = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$  of  $W_n^+$ , we define

$$\overline{\Gamma} = \langle \overline{\sigma_1}, \dots, \overline{\sigma_{n-1}} \rangle,$$

where  $\overline{\sigma_i}$  is the image of  $\overline{s_i}$ .

The automorphism group of a chiral polytope is a string  $C^+$ -group. Similarly, the rotation subgroup of a directly regular polytope is a string  $C^+$ -group. Furthermore, we can build a polytope from a string  $C^+$ -group, and the result is either chiral or directly regular. In fact, if  $\Gamma$  is a string  $C^+$ -group, then it is the automorphism group of a chiral polytope if and only if there is no automorphism of  $\Gamma$  mapping each  $\sigma_i$  to  $\overline{\sigma_i}$  (see [14, Thm. 1(c)]).

More generally, if  $\Gamma$  is a quotient of  $W_n^+$  but not a string  $C^+$ -group, then we may nevertheless build something from it that will be somewhat like a polytope, such that the notions of regularity and chirality still make sense.

#### 2.3 Smallest regular covers and mixing

If  $G = \langle g_1, \ldots, g_n \rangle$  and  $H = \langle h_1, \ldots, h_n \rangle$ , then their mix is defined as

$$G \diamond H = \{(g_1, h_1), \dots, (g_n, h_n)\} \subseteq G \times H.$$

If  $\mathcal{P}$  and  $\mathcal{Q}$  are directly regular *n*-polytopes, then we define  $\mathcal{P} \diamond \mathcal{Q}$  to be the directly regular polytope with rotation subgroup  $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ . See [4] for more details.

Every chiral polytope has a unique smallest regular cover, which may or may not be a polytope (see [12, Prop. 4.1].) If  $\mathcal{P}$  is a chiral polytope with automorphism group  $\Gamma$ , then the rotation subgroup of its smallest regular cover  $\mathcal{R}$  is  $\Lambda = \Gamma \diamond \overline{\Gamma}$ . Then  $\mathcal{R}$  will be a polytope if and only if  $\Lambda$  is a string  $C^+$ -group.

Our goal is to characterize those chiral polytopes  $\mathcal{P}$  whose smallest regular cover is a polytope. We recall the following:

**Proposition 2.2.** ([11, Cor. 7.5]) If  $\mathcal{P}$  is a chiral polytope with regular facets or vertex-figures, then its smallest regular cover is a polytope.

Thus, it suffices to concentrate on the case where  $\mathcal{P}$  has chiral facets and vertex-figures. We start with the following simple observation:

**Proposition 2.3.** Let  $\mathcal{P}$  be a chiral polytope with chiral facets and vertex-figures, with automorphism group  $\Gamma$ , and let  $\Lambda = \Gamma \diamond \overline{\Gamma}$ . Then the smallest regular cover of  $\mathcal{P}$  is a polytope if and only if  $\Lambda_0 \cap \Lambda_{n-1} \leq \Lambda_{0,n-1}$ .

*Proof.* Since the facets of the facets of a chiral polytope are regular, the facets of  $\mathcal{P}$  are chiral polytopes with regular facets. Thus, by Proposition 2.2, their smallest regular cover is a polytope. The automorphism group of this smallest regular cover is  $\Gamma_{n-1} \diamond \overline{\Gamma_{n-1}}$ , and so it is a string  $C^+$ -group. In fact, this just says that  $\Lambda_{n-1}$  is a string  $C^+$ -group. A dual argument shows that  $\Lambda_0$  is a string  $C^+$ -group. Then it follows that  $\Lambda$  is a string  $C^+$ -group if and only if  $\Lambda_0 \cap \Lambda_{n-1} = \Lambda_{0,n-1}$ , and the inclusion  $\Lambda_{0,n-1} \leq \Lambda_0 \cap \Lambda_{n-1}$  is obvious.

Finally, let us recall the notion of the chirality group of a polytope, as defined in [1]. If  $\mathcal{P}$  is a chiral polytope with automorphism group  $\Gamma$  (relative to a chosen base flag), then  $\Gamma$  is naturally isomorphic to  $W_n^+/M$  for some normal subgroup M of  $W_n^+$ . Then the *chirality group* of  $\mathcal{P}$  is the kernel of the natural projection  $\pi: W_n^+/M \to W_n^+/M\overline{M}$ .

**Proposition 2.4.** Let  $\mathcal{P}$  be a chiral polytope with automorphism group  $\Gamma$ , and let  $\Lambda = \Gamma \diamond \overline{\Gamma}$ . Then

$$X(\mathcal{P}) = \{ \gamma \in \Gamma \mid (\gamma, 1) \in \Lambda \}.$$

Proof. Note that  $(\gamma, 1) \in \Lambda$  if and only if there is some  $g = s_{i_1} \cdots s_{i_k} \in W_n^+$  such that g projects to  $\gamma$  in  $W_n^+/M$  and g projects to 1 in  $W_n^+/\overline{M}$ . In other words,  $g \in \overline{M}$  and thus  $g \in M\overline{M}$ . Thus, projecting g to  $\gamma$  in  $W_n^+/M$  and then projecting  $\gamma$  to  $W_n^+/M\overline{M}$  must yield 1, and so  $(\gamma, 1) \in \Lambda$  if and only if  $\gamma \in X(\mathcal{P})$ .

Note that if  $\mathcal{P}$  has facets isomorphic to  $\mathcal{K}$ , then  $X(\mathcal{K})$  naturally embeds into  $X(\mathcal{P})$  as:

$$X(\mathcal{K}) = \{ \gamma \in \Gamma_{n-1} \mid (\gamma, 1) \in \Lambda_{n-1} \}.$$

Similarly, if the vertex-figures are isomorphic to  $\mathcal{L}$ , then

$$X(\mathcal{L}) = \{ \gamma \in \Gamma_0 \mid (\gamma, 1) \in \Lambda_0 \}.$$

Finally, if the medial sections are isomorphic to  $\mathcal{M}$ , then

$$X(\mathcal{M}) = \{ \gamma \in \Gamma_{0,n-1} \mid (\gamma, 1) \in \Lambda_{0,n-1} \}.$$

## 3 Main result

We are now ready for the main result.

**Theorem 3.1.** Let  $\mathcal{P}$  be a chiral polytope with facets isomorphic to  $\mathcal{K}$ , vertex-figures isomorphic to  $\mathcal{L}$ , and medial sections isomorphic to  $\mathcal{M}$ . Then the smallest regular cover of  $\mathcal{P}$  is a polytope if and only if  $X(\mathcal{K}) \cap X(\mathcal{L}) \leq X(\mathcal{M})$ .

*Proof.* First, note that

$$X(\mathcal{K}) \cap X(\mathcal{L}) = \{ \gamma \in \Gamma_{n-1} \mid (\gamma, 1) \in \Lambda_{n-1} \} \cap \{ \gamma \in \Gamma_0 \mid (\gamma, 1) \in \Lambda_0 \}$$
$$= \{ \gamma \in \Gamma_0 \cap \Gamma_{n-1} \mid (\gamma, 1) \in \Lambda_0 \cap \Lambda_{n-1} \}$$
$$= \{ \gamma \in \Gamma_{0,n-1} \mid (\gamma, 1) \in \Lambda_0 \cap \Lambda_{n-1} \},$$

where the last equality follows since  $\Gamma$  is a string  $C^+$  group. Recall that

$$X(\mathcal{M}) = \{ \gamma \in \Gamma_{0,n-1} \mid (\gamma, 1) \in \Lambda_{0,n-1} \}.$$

Thus, if  $\Lambda_0 \cap \Lambda_{n-1} \leq \Lambda_{0,n-1}$ , then  $X(\mathcal{K}) \cap X(\mathcal{L}) \leq X(\mathcal{M})$ . In light of Proposition 2.3, it remains to prove the converse.

Suppose that  $(\gamma, \beta) \in \Lambda_0 \cap \Lambda_{n-1}$ . In particular,  $\beta \in \overline{\Gamma}_0 \cap \overline{\Gamma}_{n-1}$ , and since  $\overline{\Gamma}$  is a string  $C^+$ -group, it follows that  $\beta \in \overline{\Gamma}_{0,n-1}$ . Then there is an element  $\alpha$  of  $\Lambda_{0,n-1}$  such that  $(\gamma, \beta)\alpha = (\gamma', 1)$  for some  $\gamma'$ . Furthermore,  $(\gamma', 1) \in \Lambda_0 \cap \Lambda_{n-1}$ , from which it follows that  $\gamma' \in \Gamma_{0,n-1}$ . Thus  $(\gamma', 1) \in X(\mathcal{K}) \cap X(\mathcal{L})$  which, by assumption, is contained in  $X(\mathcal{M})$ . So  $(\gamma', 1) \in \Lambda_{0,n-1}$ , and since  $\alpha \in \Lambda_{0,n-1}$ , so is  $(\gamma', 1)\alpha^{-1} = (\gamma, \beta)$ , completing the proof.  $\square$ 

The beauty of Theorem 3.1 is that it does not depend on  $\mathcal{P}$  per se – merely on the type of its facets and vertex-figures. Let us consider several examples and corollaries.

**Remark 3.2.** Note that Theorem 3.1 does not require that the facets and vertex-figures of  $\mathcal{P}$  both be chiral. If, for example, the facets  $\mathcal{K}$  are regular, then  $X(\mathcal{K})$  is trivial, and so the condition of the theorem is trivially satisfied, agreeing with Proposition 2.2.

Corollary 3.3. Suppose  $\mathcal{P}$  is a chiral polytope with automorphism group  $\Gamma$ , facets isomorphic to  $\mathcal{K}$ , and vertex-figures isomorphic to  $\mathcal{L}$ . If the medial sections of  $\mathcal{P}$  are regular, then the smallest regular cover of  $\mathcal{P}$  is a polytope if and only if  $X(\mathcal{K}) \cap X(\mathcal{L})$  (viewed as subgroups of  $\Gamma$ ) has trivial intersection with  $\Gamma_{0,n-1}$ .

Corollary 3.4. If  $\mathcal{P}$  is a chiral 4-polytope with automorphism group  $\Gamma$ , facets isomorphic to  $\mathcal{K}$ , and vertex-figures isomorphic to  $\mathcal{L}$ , then the smallest regular cover of  $\mathcal{P}$  is a polytope if and only if no power of  $\sigma_2$  is contained in  $X(\mathcal{K}) \cap X(\mathcal{L})$ .

**Example 3.5.** The five families of chiral 4-polytopes with chiral facets (and vertex-figures) in [7, Table 4] have a power of  $\sigma_2$  in  $X(\mathcal{K}) \cap X(\mathcal{L})$  (see [7, Prop. 11]), and so the smallest regular cover of each one is not a polytope.

A chiral polytope  $\mathcal{P}$  is called *totally chiral* if  $X(\mathcal{P}) = \Gamma$  (see [1]). The following result is immediate.

**Proposition 3.6.** If  $\mathcal{P}$  has totally chiral facets and vertex-figures, then its smallest regular cover is a polytope if and only if it has totally chiral medial sections.

Finally, we note that Theorem 3.1 guarantees that certain choices of facets for a chiral polytope will guarantee that the smallest regular cover is a polytope.

Corollary 3.7. Suppose that K is a chiral polytope with automorphism group  $\Gamma$ . If X(K) has trivial intersection with  $\Gamma_0$ , then the smallest regular cover of every chiral polytope with facets isomorphic to K is a polytope.

**Example 3.8.** The chiral polyhedra  $\{4,4\}_{(b,c)}$  with  $bc(b-c) \neq 0$  all have chirality groups that trivially intersect  $\langle \sigma_2 \rangle$ . (Indeed, all such polyhedra cover the regular map  $\{4,4\}_{(1,0)}$ , and so no power of  $\sigma_2$  can be present in the chirality group.) Thus, all chiral polytopes with such facets have a smallest regular cover that is a polytope.

# 4 Minimal regular covers

A regular polytope  $\mathcal{R}$  is said to be a *minimal regular cover* of the polytope  $\mathcal{P}$  if the only regular cover of  $\mathcal{P}$  that is covered by  $\mathcal{R}$  is  $\mathcal{R}$  itself. Whenever the smallest regular cover of  $\mathcal{P}$  is a polytope, then this is the unique minimal regular cover of  $\mathcal{P}$  [11, Prop. 3.16]. Otherwise, a polytope may have multiple minimal regular covers; for example, see the Tomotope [10]. Indeed, it seems likely that every polytope whose smallest regular cover is not a polytope

has multiple minimal regular covers. Problem 4 in [13] asks for a characterization of chiral polytopes with a unique minimal regular cover.

Let us look at an example inspired by Example 3.5. Consider the chiral polytope  $\mathcal{P}$  whose automorphism group is

$$\Gamma = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^6 = \sigma_2^9 = \sigma_3^6 = 1, (\sigma_1 \sigma_2)^2 = (\sigma_2 \sigma_3)^2 = (\sigma_1 \sigma_2 \sigma_3)^2 = 1, \sigma_2 \sigma_1^2 = \sigma_1^2 \sigma_2^4, \sigma_3^2 \sigma_2 = \sigma_2^4 \sigma_3^2 \rangle.$$

This is a polytope with 648 flags, and its smallest regular cover  $\mathcal{R}$ , with 1944 flags, is not a polytope. Clearly, any regular polytope  $\mathcal{Q}$  that covers  $\mathcal{R}$  such that  $|\mathcal{Q}|/|\mathcal{R}|$  is a prime number will be a minimal regular cover of  $\mathcal{P}$ .

Now, let  $\mathcal{K}$  be a facet of  $\mathcal{R}$ , and let  $\mathcal{T}$  be its trivial extension, which has rotation subgroup

$$\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^6 = \sigma_2^9 = \sigma_3^2 = 1, (\sigma_1 \sigma_2)^2 = (\sigma_2 \sigma_3)^2 = (\sigma_1 \sigma_2 \sigma_3)^2 = 1, (\sigma_1 \sigma_2^{-2})^2 = \sigma_1 \sigma_2^2 \sigma_1 (\sigma_1 \sigma_2^{-1} \sigma_1)^2 = 1 \rangle.$$

Let  $Q = \mathcal{R} \diamond \mathcal{T}$ . Using the GAP package RAMP [3, 8], we confirmed that Q is a regular polytope with 5832 flags, making it a 3-fold cover of  $\mathcal{R}$  and thus a minimal regular cover of  $\mathcal{P}$ . Similarly, if we mix  $\mathcal{R}$  with the dual of  $\mathcal{T}$ , which has group

$$\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2 = \sigma_2^9 = \sigma_3^6 = 1, (\sigma_1 \sigma_2)^2 = (\sigma_2 \sigma_3)^2 = (\sigma_1 \sigma_2 \sigma_3)^2 = 1, (\sigma_3^{-1} \sigma_2^2)^2 = \sigma_2 \sigma_3^2 \sigma_2 (\sigma_3^{-1} \sigma_2 \sigma_3^{-1})^2 = 1 \rangle,$$

then we obtain another 3-fold cover of  $\mathcal{R}$ , and these two covers are non-isomorphic. So  $\mathcal{P}$  does not have a unique minimal regular cover. We conjecture that the situation is similar with all of the chiral polytopes mentioned in Example 3.5.

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